Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Wednesday 13 October 2021 – Afternoon

Time 2 hours

Paper reference **9MA0/02**

Mathematics

Advanced
PAPER 2: Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information
- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 15 questions in this question paper. The total mark for this paper is 100.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

Pearson

1. In an arithmetic series

- the first term is 16
- the 21st term is 24

(a) Find the common difference of the series.

(2)

(b) Hence find the sum of the first 500 terms of the series.

(2)

a) arithmetic series → common difference between all terms

$a_n = a_1 + (n-1)d$

first term — number of the term — common difference

$a_n = a_1 + (n-1)d$

$24 = 16 + (21-1)d$

$8 = 20d$

$\boxed{d = 0.4}$

b) $S_n = \frac{1}{2}n\{2a_1 + (n-1)d\}$

sum of n terms — number of terms — first term

$S_n = \frac{1}{2}n\{2a_1 + (n-1)d\}$

$= \frac{1}{2} \times 500\{2 \times 16 + (500-1) \times 0.4\}$

$= 250(32 + (499) \times 0.4)$

$= \boxed{57900}$

Question 1 continued

(Total for Question 1 is 4 marks)

2. The functions f and g are defined by

$$f(x) = 7 - 2x^2 \qquad x \in \mathbb{R}$$

$$g(x) = \frac{3x}{5x-1} \qquad x \in \mathbb{R} \qquad x \neq \frac{1}{5}$$

(a) State the range of f
(1)

(b) Find gf(1.8)
(2)

(c) Find $g^{-1}(x)$
(2)

a) $f(x) = 7 - 2x^2$ $\quad$ $2x^2 \geqslant 0$ for all values of x, so the greatest value of $f(x)$ is 7

$\boxed{y \leqslant 7}$

b) $f(1.8) = 7 - 2(1.8^2)$
$= 7 - 6.48$
$= 0.52$

← start with f(1.8) because we work from the inside to the outside.

$g(0.52) = \dfrac{3(0.52)}{5(0.52) - 1}$
$= \dfrac{1.56}{2.6 - 1}$
$= 0.975$

$\boxed{gf(1.8) = 0.975}$

c) To find $g^{-1}(x)$, swap the x values for y and vice versa, then solve for y.

$y = \dfrac{3x}{5x-1}$

swap the xs and ys

$x = \dfrac{3y}{5y-1}$

Question 2 continued

$$5xy - x = 3y$$

$$5xy - 3y = x$$

$$y(5x - 3) = x$$

$$y = \frac{x}{5x-3}$$

$$\boxed{g^{-1}(x) = \frac{x}{5x-3}}$$

(Total for Question 2 is 5 marks)

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3. Using the laws of logarithms, solve the equation

$$\log_3 (12y + 5) - \log_3 (1 - 3y) = 2$$

(3)

$\log_3(12y+5) - \log_3(1-3y) = 2$

$\boxed{\log_a x - \log_a y = \log_a \frac{x}{y}}$

$\log_3\left(\frac{12y+5}{1-3y}\right) = 2$ ← $\log_3 9 = 2$ because $3^2 = 9$

giving both sides the same base means we can equate $\frac{12y+5}{1-3y}$ and 9

$\log_3\left(\frac{12y+5}{1-3y}\right) = \log_3 9$

$\frac{12y+5}{1-3y} = 9$

rearrange and solve for y

$12y + 5 = 9 - 27y$

$39y = 4$

$\boxed{y = \frac{4}{39}}$

Question 3 continued

(Total for Question 3 is 3 marks)

4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$4\sin\frac{\theta}{2} + 3\cos^2\theta \approx a + b\theta + c\theta^2$$

where a, b and c are integers to be found.

(3)

$\sin\theta \approx \theta$

$\cos\theta = 1 - \frac{\theta^2}{2}$

$4\sin\frac{\theta}{2} + 3\cos^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\approx 4\sin\frac{\theta}{2} + 3(1-\sin^2\theta)$

using small angle approximation, $\sin\left(\frac{\theta}{2}\right) = \frac{\theta}{2}$

$\sin^2\theta = (\sin\theta)(\sin\theta) = (\theta)(\theta) = \theta^2$

using small angle approximation, $\sin\theta \approx \theta$, $\sin^2\theta = \theta^2$

$\approx 4\left(\frac{\theta}{2}\right) + 3(1-\theta^2)$

$\approx 2\theta + 3(1-\theta^2)$

$\approx 2\theta + 3 - 3\theta^2$

$\boxed{\approx 3 + 2\theta - 3\theta^2}$

Question 4 continued

(Total for Question 4 is 3 marks)

5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

a)

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11$$

$$\frac{dy}{dx} = 4(5x^3) - 3(24x^2) + 2(42x) - 32$$

$$= \boxed{20x^3 - 72x^2 + 84x - 32}$$

find $\frac{dy}{dx}$ by bringing down the power, and subtracting 1 from the power.

$$\frac{d^2y}{dx^2} = 3(20x^2) - 2(72x) + 84$$

$$= \boxed{60x^2 - 144 + 84}$$

to find $\frac{d^2y}{dx^2}$, just differentiate again, bringing the power down and subtracting 1 from the power.

b)i) Because the key word is verify, just sub in $x=1$ into $\frac{dy}{dx}$ and conclude.

$$\frac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$$

a stationary point occurs when the gradient $= 0$, so $\frac{dy}{dx} = 0$.

@ $x=1$ $\quad \frac{dy}{dx} = 20(1^3) - 72(1^2) + 84(1) - 32$

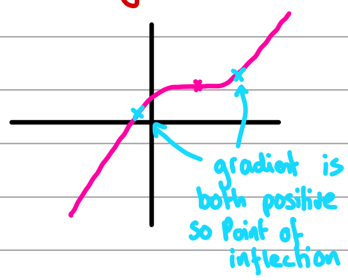
$$= 20 - 72 + 84 - 32$$

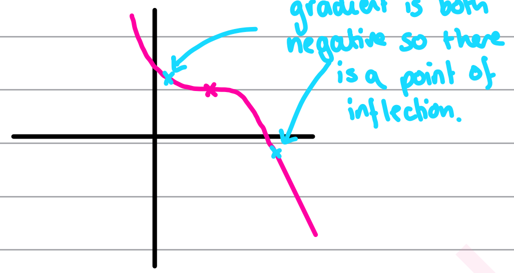
$$= 0$$

$\boxed{\text{At } x=1 \ \frac{dy}{dx} = 0 \text{, so there is a stationary point at } x=1.}$

Question 5 continued

ii) To verify a point of inflection, find the gradient of two points either side of $x=1$. If their gradients are either both positive or both negative, it is a point of inflection.





$@x=0.9 \quad \frac{dy}{dx} = 20(0.9)^3 - 72(0.9)^2 + 84(0.9) - 32$

$= -0.14$

$@x=1.1 \quad \frac{dy}{dx} = 20(1.1)^3 - 72(1.1)^2 + 84(1.1) - 32$

$= -0.1$

$-0.14 < 0$

$-0.1 < 0$

A $x=0.9$ and $x=1.1$, the gradients are both negative, so $x=1$ is a point of inflection.

(Total for Question 5 is 7 marks)

6.

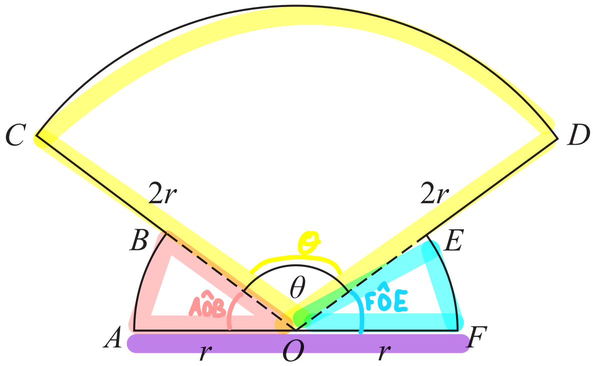


Figure 1

The shape $OABCDEFO$ shown in Figure 1 is a design for a logo.

In the design
- OAB is a sector of a circle centre O and radius r
- sector OFE is congruent to sector OAB
- ODC is a sector of a circle centre O and radius $2r$
- AOF is a straight line

Given that the size of angle COD is θ radians,

(a) write down, in terms of θ, the size of angle AOB **(1)**

(b) Show that the area of the logo is

$$\frac{1}{2}r^2(3\theta + \pi)$$

(2)

(c) Find the perimeter of the logo, giving your answer in simplest form in terms of r, θ and π. **(2)**

a) Angle $A\hat{O}F = \pi$ radians (180° = π radians)

$A\hat{O}F = A\hat{O}B + F\hat{O}E + \theta = \pi$

the two smaller sectors are congruent, so the angles AÔB and FÔE are the same. → $2A\hat{O}B + \theta = \pi$

$2A\hat{O}B = \pi - \theta$

$$\boxed{A\hat{O}B = \frac{\pi - \theta}{2}}$$

b) area of sector $= \frac{1}{2}\theta r^2$

use angle $= \frac{\pi - \theta}{2}$ we found in part a

area of logo = area OAB + area ODC + area OFE

$$= \frac{1}{2}\left(\frac{\pi-\theta}{2}\right)r^2 + \frac{1}{2}\theta(2r)^2 + \frac{1}{2}\left(\frac{\pi-\theta}{2}\right)r^2$$

area OFE = OAB →

$$= \left(\frac{\pi-\theta}{2}\right)r^2 + \frac{1}{2}\theta(2r)^2$$

Simplify to get into the required form

$$= \left(\frac{\pi-\theta}{2}\right)r^2 + 2\theta r^2$$

$$= r^2\left(\frac{\pi-\theta}{2} + 2\theta\right)$$

$$= r^2\left(\frac{\pi+3\theta}{2}\right)$$

$$\boxed{= \frac{1}{2}r^2(\pi + 3\theta)}$$

c) arc length $= \theta r$

perimeter = OA + AB + BC + CD + DE + EF + FO

OA = OF
AB = EF
BC = DE →

$$= 2(OA) + 2(AB) + 2(BC) + CD$$

OA = r

use arc length $= \theta r$, $\theta = \frac{\pi-\theta}{2}$

BC = r

use arc length $= \theta r$, $r = 2r$

$$= 2(r) + 2\left(r\left(\frac{\pi-\theta}{2}\right)\right) + 2(r) + 2r\theta$$

$$= 4r + r(\pi-\theta) + 2r\theta$$

$$= r(4 + \pi - \theta + 2\theta)$$

$$\boxed{= r(4 + \pi + \theta)}$$

Question 6 continued

Question 6 continued

(Total for Question 6 is 5 marks)

7. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

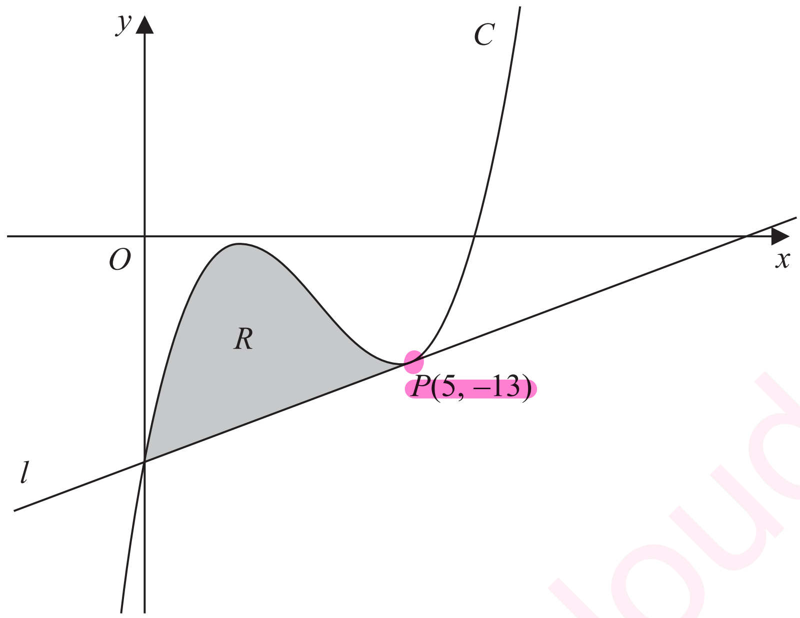


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

(b) Hence verify that l meets C again on the y-axis.

(1)

The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R.

(4)

a) Differentiate C

$y = x^3 - 10x^2 + 27x - 23$

$\frac{dy}{dx} = 3(x^2) - 2(10x) + 27$

$\frac{dy}{dx} = 3x^2 - 20x + 27$

differentiate C by bringing the power down and subtracting 1 from the power.

Find gradient at (5, -13):

@ $x = 5$ $\quad \frac{dy}{dx} = 3(5)^2 - 20(5) + 27$

$= 2$

Sub in $x = 5$ to find the gradient of C at (-5, 13)

gradient of C at $(5, -13) = 2$

$\therefore$ gradient of $L = 2$ (because L is a tangent to C)

Find equation of L using $y = mx + c$ and subbing in (5, -13) and gradient = 2

$y = mx + c$

$y = 2x + c$

$-13 = 2(5) + c$

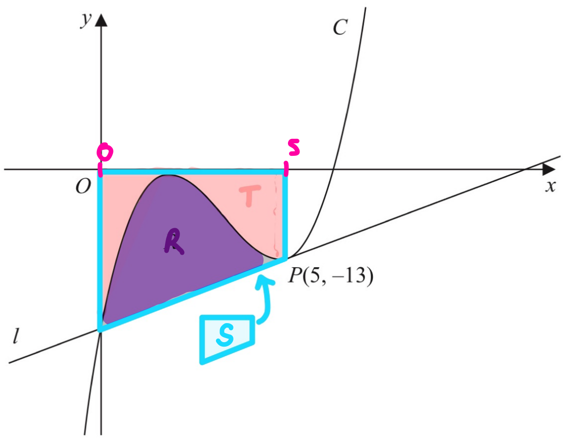
$c = -23$

L: $\boxed{y = 2x - 23}$

b) Both c and L pass through (0, -23), so C meets L again on the y axis

Question 7 continued

c)



Using area of a trapezium to find S:

$$S = \frac{1}{2}(a+b)h$$

ie half the sum of the parallel sides, times the distance between them

$-T$, because T is below the x axis and so will be negative, so we need to subtract the magnitude of T from S.

$$R = S - -T$$

$$= \left(\frac{1}{2}\times(13+23)\times 5\right) - -\int_0^5 x^3 - 10x^2 + 27x - 23$$

integrate by adding 1 to the power, and dividing by the new power

$$= 90 + \left[\frac{1}{4}x^4 - \frac{10}{3}x^3 + \frac{27}{2}x^2 - 23x\right]_0^5$$

$$= 90 + \left(\frac{1}{4}(5)^4 - \frac{10}{3}(5)^3 + \frac{27}{2}(5)^2 - 23(5)\right) - (0)$$

$$= 90 + \left(-\frac{455}{12}\right)$$

$$\boxed{= \frac{625}{12}}$$

Question 7 continued

(Total for Question 7 is 9 marks)

8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

use the product rule : ie, differentiate x then multiply by qx, then differentiate y $(=\frac{dy}{dx})$ and multiply by qx

To do implicit differentiation:

↳ differentiate every term with respect to x

↳ Multiply every term where y is differentiated by $\frac{dy}{dx}$

↳ solve to find $\frac{dy}{dx}$

$$px^3 + qxy + 3y^2 = 26$$

$$3px^2 + qy + qx\left(\frac{dy}{dx}\right) + 6y\left(\frac{dy}{dx}\right) = 0$$

$$qx\left(\frac{dy}{dx}\right) + 6y\left(\frac{dy}{dx}\right) = -3px^2 - qy$$

$$\frac{dy}{dx}\left(qx + 6y\right) = -3px^2 - qy$$

$$\boxed{\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}}$$

b) Sub in $(-1, -4)$ into the equation C to find an equation in terms of p and q.

$$px^3 + qxy + 3y^2 = 26$$

@$P(-1,-4)$ $\quad p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$

$$-p + 4q + 48 = 26$$

$$4q - p + 22 = 0$$

Question 8 continued

Find the gradient of the normal to C at P

$$19x + 26y + 123 = 0$$
$$26y = -19x - 123$$
$$y = -\frac{19}{26}x - 123$$

gradient of the normal to C at P $= -\frac{19}{26}$

$\therefore$ gradient of tangent to C at P: $\frac{26}{19}$ ← because the gradients of perpendicular lines multiply to give -1. $-\frac{19}{26} \times \frac{26}{19} = -1$

Use part a) to find a second equation in p and q

$$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$$

@ gradient $= \frac{26}{19}$
$P = (-1, -4)$

$$\frac{26}{19} = \frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)}$$

$$\frac{26}{19} = \frac{-3p + 4q}{-q - 24}$$

$$26(-q - 24) = 19(-3p + 4q)$$

$$-26q - 624 = -57p + 76q$$

$$57p - 102q = 624$$

Solve the two equations simultaneously to find p and q

1: $4q - p = -22$
2: $57p - 102q = 624$

57×1 $\quad 228q - 57p = -1254$

Question 8 continued

$57 \times ① + ②$ $\quad 228q - 57p + 57p - 102q = -1254 + 624$

$126q = -630$

$q = -5$

① $\quad 4(-5) - p = -22$ ← sub $q = -5$ into equation ① to find p

$-20 - p = -22$

$p = 2$

$\boxed{p = 2 \quad q = -5}$

Question 8 continued

(Total for Question 8 is 9 marks)

9. Show that

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{9}{28}$$

(3)

$S_\infty = \frac{a}{1-r}$

a = 1st term
r = common ratio

Sub in $n=2$ to find a, the first term

@ $n=2$ $\quad a = \left(\frac{3}{4}\right)^2 \cos(180\times 2)$

$= \frac{9}{16} \times \cos 360$ ← $\cos 360 = 1$

$a = \frac{9}{16}$

Because $\cos(180n)$ follows the pattern $1, -1, 1, -1, 1, -1 \ldots\ldots$, $\cos(180n)$ changes by a ratio of -1 every time.

$r = \frac{3}{4} \times -1$ ← $\left(\frac{3}{4}\right)^n$ has a common ratio of $\frac{3}{4}$, so the overall common ratio is $\frac{3}{4} \times -1$

$= -\frac{3}{4}$

Sub into $S_\infty = \frac{a}{1-r}$

$S_\infty = \dfrac{\frac{9}{16}}{1-\left(-\frac{3}{4}\right)}$

$= \dfrac{\frac{9}{16}}{\frac{7}{4}}$

$= \boxed{\frac{9}{28}}$

Question 9 continued

(Total for Question 9 is 3 marks)

10. The time, T seconds, that a pendulum takes to complete one swing is modelled by the formula

$$T = al^b$$

where l metres is the length of the pendulum and a and b are constants.

(a) Show that this relationship can be written in the form

$$\log_{10} T = b\log_{10} l + \log_{10} a$$

(2)

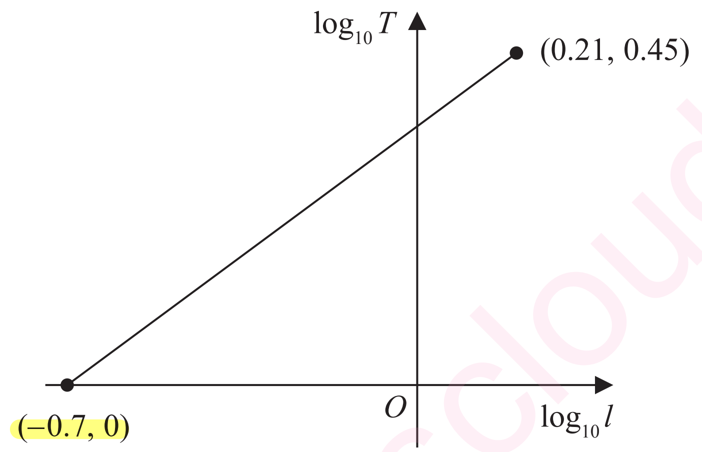


Figure 3

A student carried out an experiment to find the values of the constants a and b.

The student recorded the value of T for different values of l.

Figure 3 shows the linear relationship between $\log_{10} l$ and $\log_{10} T$ for the student's data.
The straight line passes through the points (–0.7, 0) and (0.21, 0.45)

Using this information,

(b) find a complete equation for the model in the form

$$T = al^b$$

giving the value of a and the value of b, each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant a.

(1)

a) $T = al^b$

take $\log_{10}$ of both sides

$\log_{10} T = \log_{10} al^b$

$\log_{10} T = \log_{10} a + \log_{10} l^b$

$\boxed{\log_a xy = \log_a x + \log_a y \quad \text{so } \log_{10} al^b = \log_{10} a + \log_{10} l^b}$

Question 10 continued

$\log_{10}T = \log_{10}a + b\log_{10}L$

$\log_a x^b = b\log_a x$

$\therefore \log_{10}L^b = b\log_{10}L$

$$\boxed{\log_{10}T = b\log_{10}L + \log_{10}a}$$

b) $\log_{10}T = b\log_{10}L + \log_{10}a$ is in the equation $y = mx + c$. So find the equation of the line on the graph, and $m = b$, and $c = \log_{10}a$

$$\text{gradient} = \frac{y^1 - y^2}{x^1 - x^2} = \frac{0 - 0.45}{-0.7 - 0.21} = \frac{45}{91}$$

$$y = \frac{45}{91}x + c$$

@ $(-0.7, 0)$ $\quad 0 = \frac{45}{91} \times -0.7 + c$ ← sub in the co-ordinates of any point on the line to find the value of c.

$$c = \frac{9}{26}$$

$$y = \frac{45}{91}x + \frac{9}{26}$$

$m = b$, $\therefore b = \frac{45}{91} = 0.495$ (3sf)

$c = \log_{10}a$ $\therefore \log_{10}a = \frac{9}{26}$

$$a = 10^{\frac{9}{26}}$$

$$a = 2.22 \text{ (3sf)}$$

$T = aL^b \longrightarrow \boxed{T = 2.22L^{0.495}}$

c) $\boxed{a \text{ is the time taken for one swing of a pendulum of length 1m}}$

(because when $L = 1$, $L^b = 1$ $\therefore T = a \times 1$ $\therefore T = a$)

Question 10 continued

Question 10 continued

(Total for Question 10 is 6 marks)

11.

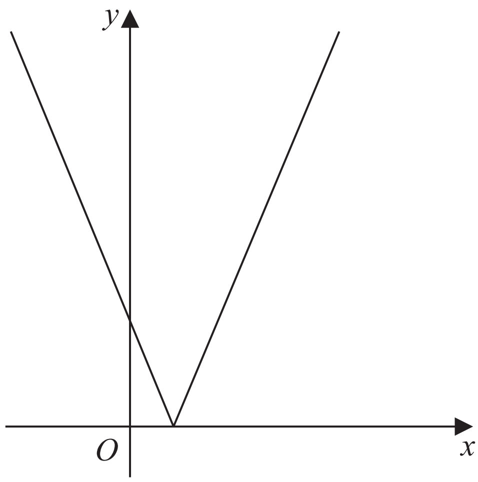


Figure 4

Figure 4 shows a sketch of the graph with equation

$$y = |2x - 3k|$$

where k is a positive constant.

the k is the y co-ordinate of the maximum point

the - sign means the graph is flipped upside down

(a) Sketch the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = k - |2x - 3k|$$

stating

- the coordinates of the maximum point
- the coordinates of any points where the graph cuts the coordinate axes

(4)

(b) Find, in terms of k, the set of values of x for which

$$k - |2x - 3k| > x - k$$

giving your answer in set notation.

(4)

(c) Find, in terms of k, the coordinates of the minimum point of the graph with equation

$$y = 3 - 5\mathrm{f}\left(\frac{1}{2}x\right)$$

(2)

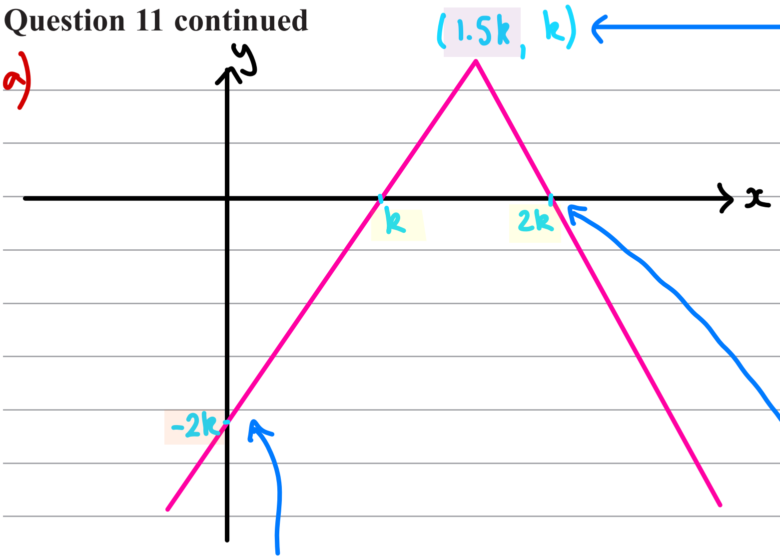
$f(\tfrac{1}{2}x) = k - |x - 3k|$

$3 - 5f(\tfrac{1}{2}x) = 3 - 5(k - |x-3k|)$

$= 3 - 5k + 5|x-3k|$

Question 11 continued

a)



Sub in $y=k$ to find the x co-ordinate of the maximum point

$y = k - |2x - 3k|$
$k = k - |2x - 3k|$
$0 = -|2x - 3k|$
$0 = 2x - 3k$
$2x = 3k$
$x = 1.5k$

Sub in $y=0$ to find the intercepts of the x axis

$0 = k - |2x - 3k|$
$|2x - 3k| = k$

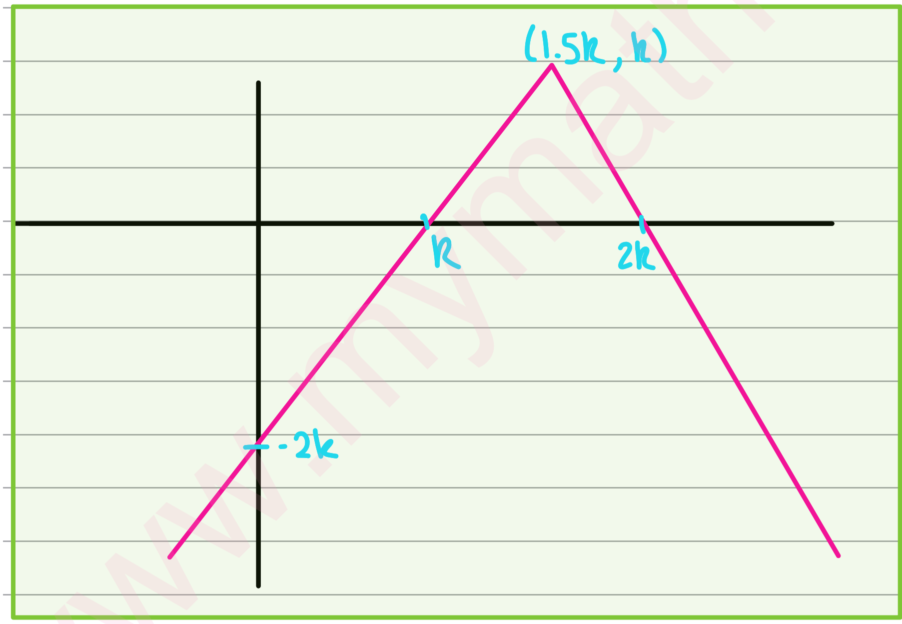
$2x - 3k = k$
$2x = 4k$
$x = 2k$

$-2x + 3k = k$
$-2x = -2k$
$x = k$

sub in $x=0$ to find the y intercept

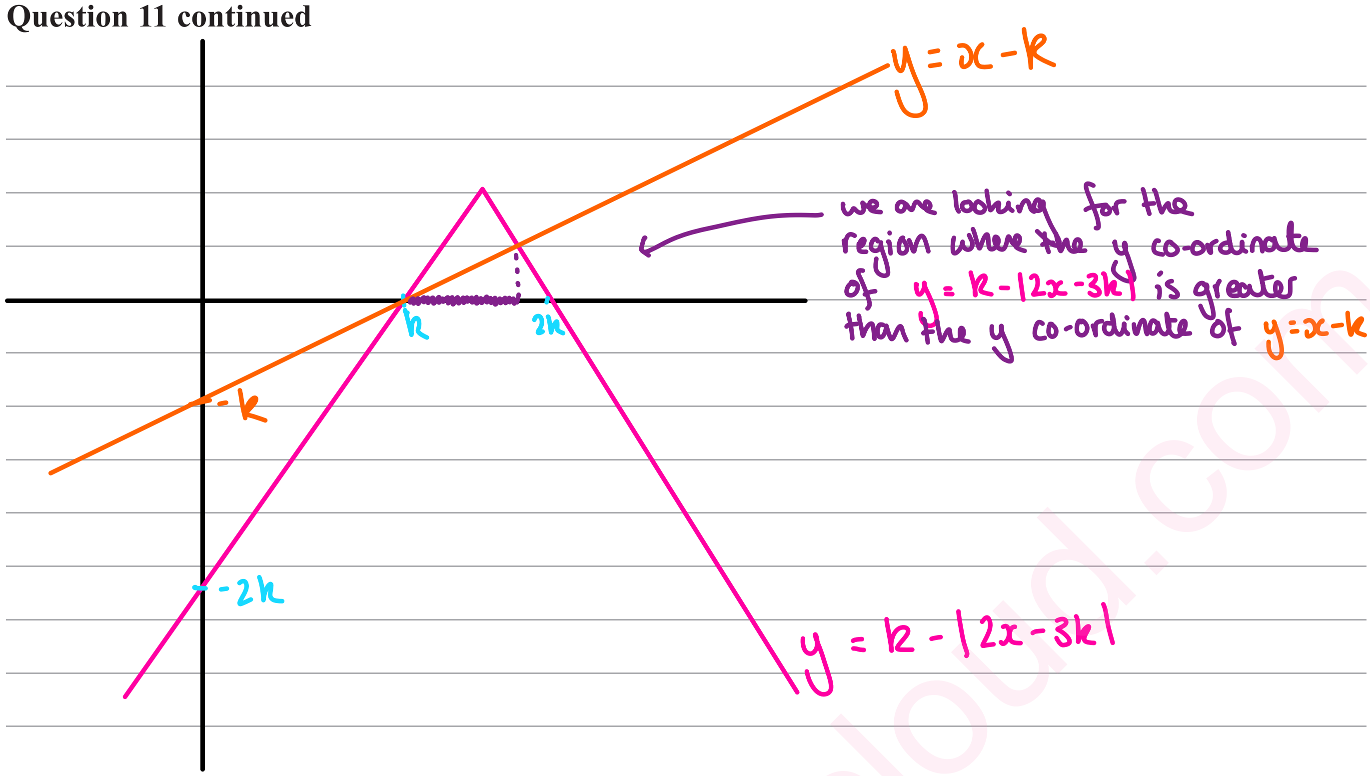
$y = k - |2x - 3k|$
$y = k - |-3k|$
$y = k - 3k$
$y = 2k$

k is positive so $|-3k| = 3k$



Turn over

Question 11 continued



$$k - |2x - 3k| = x - k$$
$$|2x - 3k| = -x + 2k$$

$2x - 3k = -x + 2k$ $3k - 2x = -x + 2k$

$3x = 5k$ $-x = -k$

$x = \frac{5}{3}k$ $x = k$

In the region we are looking for (shaded purple), $x > k$ and $x < \frac{5}{3}k$

In set notation: $\boxed{\{x : x < \frac{5k}{3}\} \cap \{x : x > k\}}$

$\cap$ means and

c) $y = 3 - 5f(\frac{1}{2}x)$

max point of $f(x) = (1.5k, k)$

max point of $f(\frac{1}{2}x) = (3k, k)$ (horizontal stretch with scale factor 2, so x co-ordinate is multiplied by 2)

P 6 8 7 3 2 A 0 3 2 4 8

Question 11 continued

minimum point of $-5f\left(\frac{1}{2}\right) = (3k, -5k)$ (reflection in x axis and vertical stretch scale factor 5, so the y co-ordinate is multiplied by -5)

minimum point of $3-5f\left(\frac{1}{2}\right) = (3k, 3-5k)$ (translation $\begin{pmatrix}0\\3\end{pmatrix}$, so add 3 to the y co-ordinate)

$\boxed{(3k, 3-5k)}$

(Total for Question 11 is 10 marks)

12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_p^q \frac{2(u-1)^3}{u}\,du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = A - B\ln 5$$

where A and B are constants to be found.

(4)

a) Find dx:

let $u = 1 + \sqrt{x}$

$\sqrt{x} = u - 1$

$x = (u-1)^2$

use the chain rule

$\frac{dx}{du} = 2(u-1)$

$dx = 2(u-1)\,du$

Find the new limits

$u = 1 + \sqrt{x}$

@$x=16$ $\quad u = 1 + \sqrt{16}$

$= 5$

@$x=0$ $\quad u = 1 + \sqrt{0}$

$= 1$ $\quad$ limits $= 1, 5$

Substitute in to eliminate x

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_1^5 \frac{(u-1)^2}{u} \times 2(u-1)\,du$$

$$= \boxed{\int_1^5 \frac{2(u-1)^3}{u}\,du}$$

b) Integrate $\int_0^5 \frac{2(u-1)^3}{u}\,du$

$$\int_1^5 \frac{2(u-1)^3}{u}\,du$$

$$= 2\int_1^5 \frac{(u-1)^3}{u}\,du$$

use binomial expansion

$(u-1)^3$
$= u^3 + 3u^2(-1) + 3(u)(-1)^2 + (-1)^3$
$= u^3 - 3u^2 + 3u - 1$

$$= 2\int_1^5 \frac{u^3 - 3u^2 + 3u - 1}{u}\,du$$

$$= 2\int_1^5 u^2 - 3u + 3 - \frac{1}{u}\,du$$

integrate by adding 1 to the power and dividing by the new power.

$\boxed{\int \frac{1}{u} = \ln u}$

$$= 2\left[\tfrac{1}{3}u^3 - \tfrac{3}{2}u^2 + 3u - \ln u\right]_1^5$$

sub in limits 5 and 1

$$= 2\left(\left(\tfrac{1}{3}(5)^3 - \tfrac{3}{2}(5)^2 + 3(5) - \ln(5)\right) - \left(\tfrac{1}{3} - \tfrac{3}{2} + 3 - \ln 1\right)\right)$$

$\ln 1 = 0$ because $e^0 = 1$

$$= 2\left(\frac{115}{6} - \ln 5 - \frac{11}{6} + 0\right)$$

$$= 2\left(\frac{104}{6} - \ln 5\right)$$

$$= \boxed{\frac{104}{3} - 2\ln 5}$$

Question 12 continued

P 6 8 7 3 2 A 0 3 6 4 8

Question 12 continued

(Total for Question 12 is 7 marks)

13. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3 \theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ **(3)**

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$ **(3)**

a) $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

find $\frac{dx}{d\theta}$, then find the reciprocal, which $= \frac{d\theta}{dx}$

$y = (\operatorname{cosec}\theta)^3$

use chain rule. Differentiate the bracket, then multiply by the derivative of the bracket.

$\frac{dy}{d\theta} = 3(\operatorname{cosec}\theta)^2 \times -\operatorname{cosec}\theta \cot\theta$

$= -3(\operatorname{cosec}\theta)^3 \cot\theta$

$\frac{dy}{d\theta} = -3\operatorname{cosec}^3\theta \cot\theta$

In formula booklet: $f'(\operatorname{cosec}\theta) = -\operatorname{cosec}\theta\cot\theta$

$x = \sin 2\theta$

$f'(\sin k\theta) = k\cos k\theta$

$\frac{dx}{d\theta} = 2\cos 2\theta$

$\frac{d\theta}{dx} = \frac{1}{2\cos 2\theta}$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= -3\operatorname{cosec}^3\theta\cot\theta \times \frac{1}{2\cos 2\theta}$

$= \boxed{\frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}}$

b) find the value of θ when $y = 8$, then sub into $\frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$ to find the gradient

$y = \operatorname{cosec}^3\theta$

@$y=8$ $\quad 8 = \operatorname{cosec}^3\theta$

Question 13 continued

$$8 = \frac{1}{\sin^3\theta}$$

$\downarrow$ $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\therefore$ $\operatorname{cosec}^3\theta = \frac{1}{\sin^3\theta}$

$$\sin^3\theta = \frac{1}{8}$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

Sub $\theta = \frac{\pi}{6}$ into $\frac{dy}{dx}$ $\frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$ to find the gradient

$$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)}{2\cos\left(\frac{\pi}{3}\right)}$$

$\cot\frac{\pi}{6} = \frac{\cos\pi/6}{\sin\pi/6} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

from the question, $\operatorname{cosec}^3\theta = 8$

$$= \frac{-3 \times 8 \times \sqrt{3}}{2 \times 0.5}$$

$$= \boxed{-24\sqrt{3}}$$

(Total for Question 13 is 6 marks)

14.

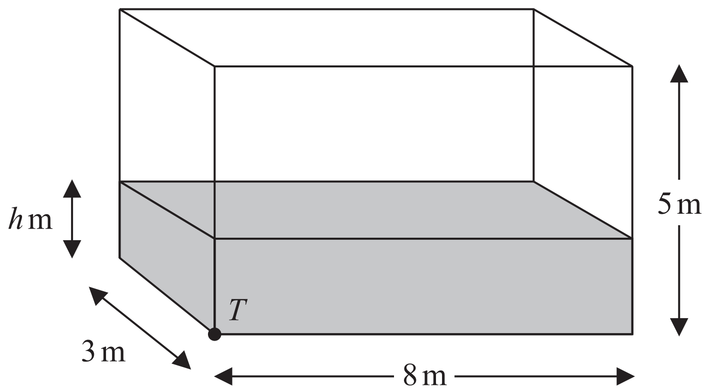


Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

a) $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $\longrightarrow$ so we need to find $\frac{dh}{dV}$ and $\frac{dV}{dt}$

$\frac{dV}{dt} = 0.48 - 0.1h$ (water flowing into the tank - water flowing out of the tank)

$\frac{dV}{dt} = 0.48 - 0.1h$

$\text{Volume} = 8 \times 3 \times h$ ← container is a cuboid so find volume by multiplying the side lengths together

$V = 24h$

$\frac{dV}{dh} = 24$ ← find derivative by bringing down the power (of h) which is 1, and subtracting 1 from the power $\therefore h \to 0$

$\frac{dh}{dV} = \frac{1}{24}$ ← $\frac{dh}{dV} = \frac{1}{\frac{dV}{dh}}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{24} \times (0.48 - 0.1h)$

$\frac{dh}{dt} = \frac{0.48 - 0.1h}{24}$

$24\frac{dh}{dt} = 0.48 - 0.1h$

$\boxed{1200\frac{dh}{dt} = 24 - 5h}$

Integrate $1200\frac{dh}{dt} = 24 - 5h$ to find h

$1200\frac{dh}{dt} = 24 - 5h$

put all the h values on one side and all t values on the other side

$\frac{1200}{24-5h}\,dh = dt$

because the power of h is 1 higher on the bottom than it is on the top, use ln

$1200\int \frac{1}{24-5h}\,dh = \int dt$

$1200 \times -\frac{1}{5}\ln(24-5h) = t + c$

$-240\ln(24-5h) = t + c$

To find c:
at $t = 0$, $h = 2$

$-240\ln(24 - 5(2)) = c$

Question 14 continued

$c = -240\ln(14)$

Sub in c

$-240\ln(24-5h) = t - 240\ln(14)$

$a\ln x - a\ln y = a\ln\left(\frac{x}{y}\right)$

$t = 240\ln(14) - 240\ln(24-5h)$

$t = 240\ln\left(\frac{14}{24-5h}\right)$

$\frac{1}{240}t = \ln\left(\frac{14}{24-5h}\right)$

raise each side to base e

$e^{\ln\left(\frac{14}{24-5h}\right)} = \frac{14}{24-5h}$

$e^{\frac{t}{240}} = \frac{14}{24-5h}$

rearrange for h

$24-5h = \frac{14}{e^{t/240}}$

$5h = 24 - \frac{14}{e^{t/240}}$

$h = \frac{24}{5} - \frac{14}{5e^{t/240}}$

$\frac{1}{e^{t/240}} = e^{-\frac{t}{240}}$

$\boxed{h = 4.8 - 2.8e^{-t/240}}$

c) $h = 4.8 - 2.8e^{-t/240}$

As $t \to \infty$ $\quad e^{-t/240} \to 0 \quad h \to 4.8$

When t is very large, the height of the water is 4.8m. The tank is 5m high so the tank will never become full.

Question 14 continued

(Total for Question 14 is 12 marks)

15. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

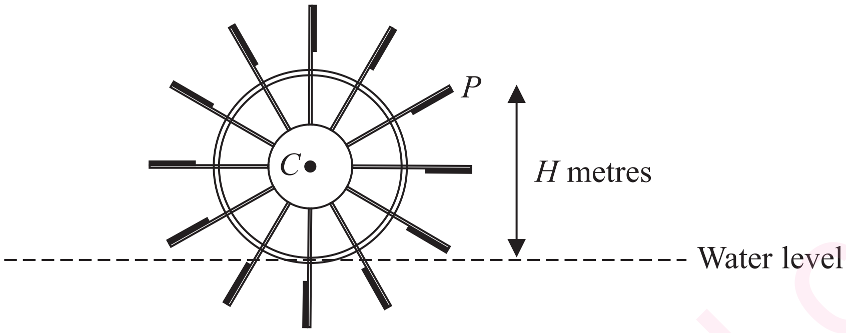


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable*.)

(4)

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account.

(1)

Question 15 continued

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

Use formula booklet

a) $2\cos\theta - \sin\theta = R\cos(\theta + \alpha)$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

compare coefficients: $2\cos\theta = R\cos\theta\cos\alpha \rightarrow 2 = R\cos\alpha$
$\sin\theta = R\sin\theta\sin\alpha \rightarrow 1 = R\sin\alpha$

find α:
$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{2}$

$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$\tan\alpha = \frac{1}{2}$
$\alpha = 0.464$ (3sf)

find R:
$R = \sqrt{2^2 + 1^2}$
$= \sqrt{5}$

$\boxed{2\cos\theta - \sin\theta = \sqrt{5}\cos(\theta + 0.464)}$

b) $H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$
$= 3 + 2(2\cos(0.5t) - \sin(0.5t))$
$= 3 + 2(\sqrt{5}\cos(0.5t + 0.464))$

sub in $\sqrt{5}\cos(\theta + 0.464)$ (that we found in part a)

$H = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$

max H occurs when $\cos(0.5t + 0.464) = 1$

$\therefore$ $\boxed{\text{max height} = 3 + 2\sqrt{5}}$

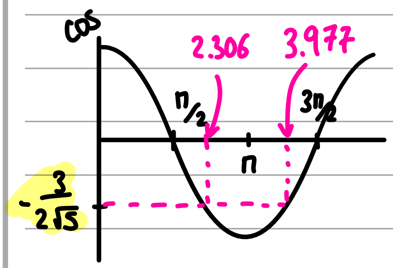
Use $\cos(0.5t + 0.464) = 1$ to find t

$\cos(0.5t + 0.464) = 1$
$0.5t + 0.464 = 2\pi$
$0.5t = 2\pi - 0.464$
$t = 4\pi - 0.928$
$\boxed{t = 11.6 \text{ seconds}}$

Question 15 continued

c) Find the times when the height is equal to 0, then find the difference between these times.

$$3 + 2\sqrt{5}\cos(0.5t + 0.464) = 0$$
$$2\sqrt{5}\cos(0.5t + 0.464) = -3$$
$$\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}}$$
$$0.5t + 0.464 = \cos^{-1}\left(\frac{-3}{2\sqrt{5}}\right)$$
$$0.5t = \cos^{-1}\left(\frac{-3}{2\sqrt{5}}\right) - 0.464$$
$$t = 2\left(\cos^{-1}\left(\frac{-3}{2\sqrt{5}}\right) - 0.464\right)$$



Find the first two values of $\cos^{-1}\left(\frac{-3}{2\sqrt{5}}\right)$

$$t_1 = 2(2.306 - 0.464)$$
$$= 3.684$$

$$t_2 = 2(3.977 - 0.464)$$
$$= 7.026$$

$$T = t_2 - t_1$$
$$= 7.026 - 3.684$$
$$\boxed{= 3.34 \text{ seconds}}$$

d) $\boxed{\text{the '3' in the equation would need to vary}}$

P 6 8 7 3 2 A 0 4 6 4 8

Question 15 continued

Question 15 continued

(Total for Question 15 is 11 marks)

TOTAL FOR PAPER IS 100 MARKS

P 6 8 7 3 2 A 0 4 8 4 8